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On the relation between two numerical methods for the computation of random surface gravity waves

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Abstract

Recently two numerical spectral methods, based on the use of the Fast Fourier Transform algorithm, have been found to be useful for studying the statistical properties of a large number of interacting random waves: the first one is known as the Higher Order Spectral (HOS) method and the second is based on the computation of the dynamical equation arising from the Hamiltonian description of surface gravity waves. Here we show analytically the relation between these two methods; more in particular, starting from the HOS approach and writing the corresponding evolution equations in spectral space, after a proper symmetrization of the coupling coefficients in the resulting integral terms, the Hamiltonian dynamical equations are recovered.
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Nowadays numerical simulation of fully nonlinear random waves has become an issue of particular relevance in the field of ocean waves. In the last 150 years a large number of simplified models (Boussinesq, Korteweg–de Vries, Nonlinear Schroedinger equations, just to mention some) have allowed a very deep physical understanding of nonlinearity in shallow and deep water. One important question to be addressed is how accurately these models reproduce the physics behind the primitive equations. The numerical simulations of the Euler equations can be used as an "experimental tool" for validating and understanding the limits of simpler models. Moreover, operational wave forecast models are based on the numerical integration of the so called "kinetic equation" [1], that describes the evolution of the spectral density function of the free surface elevation. A unique way of establishing the validity of the kinetic equation consists in performing direct numerical simulations of the primitive equations and make comparisons. The computation of deterministic fully nonlinear dynamics is also of particular relevance for establishing the probability density function of wave crests and wave heights.

In the past years a number of methods have been used for solving numerically the fully nonlinear equations (see [2,3] for a review). For the particular case of the dynamics of a large number of interacting random waves in two horizontal dimensions, here we mention the methods described in [4–9]. In the present note, we focus on two of these approaches that have furnished very interesting results concerning the long-time/large-domain numerical simulations of water waves. The first one is the Higher Order Spectral (HOS) method [4,5] and the other is based on the compu-

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tation of the Hamiltonian dynamical equations [9]. Both methods assume the fluid to be inviscid, incompressible and the flow to be irrotational; using these two approaches a number of interesting papers have recently appeared in the literature discussing the evolution of a large number of random waves characterized by realistic ocean wave spectra. In particular, interesting results have been obtained by Tanaka [10,11] who made very large numerical computations of surface gravity waves starting from JONSWAP spectra with random phases. The aim of Tanaka was to verify the Hasselmann theory for the nonlinear energy transfer in the wave spectrum. His computations were based on the HOS method. More recently Onorato et al. [12] have used the same method for studying the formation of power laws in the high wave-number tail of the wave spectrum (see also [13]). The method has also been used in [14] to study the formation of freak waves in random waves. On the other hand Pushkarev and Zakharov [15] have proposed the use of Hamiltonian dynamical equations for studying the formation of power laws in the spectrum of capillary waves. The same method has then been applied to surface gravity waves in [16,17,9]. If periodic boundary conditions are assumed, the Fast Fourier Transform (FFT) algorithm can be used in both approaches, reducing the number of operations from N^2 to $N \operatorname{Log} N$ (N is the total number of grid points). So far, these two approaches seem to be promising and suitable for long time or large domain evolution of a nonlinear wave field. Nevertheless, it should here be pointed out that some unwanted effects related to the discretization on a regular grid and the consequent lack of exact resonances might appear (see [18,6,19,20] for a discussion).

The aim of this note is to show the relationship between these two methods that may at first glance seem different. It should be mentioned that in the past similar work concerning the relation between the truncated Hamiltonian (in which the canonical variables are the surface elevation and the velocity potential calculated on the surface) and the classical perturbation approach at the mean equilibrium level of the free surface elevation and the velocity potential have been considered [21]. The problem considered in the present note is slightly different; the variables in both numerical methods described in our manuscript are the surface elevation and the velocity potential calculated on the surface. In one case the expansion is performed directly on the Hamiltonian and the evolution equations for the surface elevation and for the velocity potential are then derived; in the second method (HOS method) the expansion is performed, with a different methodology, directly on the evolution equations for the surface elevation and for the surface velocity potential.

The note is organized as follows: we will first briefly describe the Hamiltonian dynamical equations, then the HOS method and finally show the relationship between the two.

The Hamiltonian dynamical equations. Zakharov in [22] has shown that for inviscid, incompressible fluid and irrotational flow, the free surface elevation $\eta(\mathbf{x}, t)$ and the velocity potential $\psi = \phi(\mathbf{x}, \eta(\mathbf{x}, t), t)$, calculated on the free surface, are canonically conjugate variables:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{1}$$

where the Hamiltonian is the total energy of the system, given by:

$$H = \frac{1}{2} \int d\mathbf{x} \int\limits_{-h}^{\eta} (\nabla \phi)^2 \, dz + \frac{1}{2} g \int \eta^2 \, d\mathbf{x}. \tag{2}$$

Here $\mathbf{x}$ corresponds to the two horizontal coordinates, z is the vertical coordinates and h is water depth which, from now on, will be considered infinite. The horizontal integrals extend over the entire horizontal plane. In order to derive the evolution equations for the surface elevation and the velocity potential in a suitable form for numerical computations, it is necessary to expand the Hamiltonian in powers of steepness (details are contained in [23]). The resulting dynamical equations, written in Fourier space, up to fourth order in steepness, are the following:

$$\frac{\partial \hat{\eta}_{\mathbf{k}}}{\partial t} = |\mathbf{k}| \hat{\psi}_{\mathbf{k}} + \int\limits_{-\infty}^{\infty} U_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} \hat{\psi}_{\mathbf{k}_1} \hat{\eta}_{\mathbf{k}_2} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2} \, d\mathbf{k}_1 \, d\mathbf{k}_2$$

$$+ \int\limits_{-\infty}^{\infty} X_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \hat{\psi}_{\mathbf{k}_1} \hat{\eta}_{\mathbf{k}_2} \hat{\eta}_{\mathbf{k}_3} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3, \tag{3}$$

$$\frac{\partial \hat{\psi}_{\mathbf{k}}}{\partial t} = -g\hat{\eta}_{\mathbf{k}} + \int\limits_{-\infty}^{\infty} Y_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} \hat{\psi}_{\mathbf{k}_1} \hat{\psi}_{\mathbf{k}_2} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \, d\mathbf{k}_1 \, d\mathbf{k}_2$$

$$+ \int\limits_{-\infty}^{\infty} Z_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \hat{\psi}_{\mathbf{k}_1} \hat{\psi}_{\mathbf{k}_2} \hat{\eta}_{\mathbf{k}_3} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3, \tag{4}$$

where $\hat{\eta}(\mathbf{k})$ and $\hat{\psi}(\mathbf{k})$ are the Fourier transforms on the horizontal plane respectively of the surface elevation and the surface velocity potential. Furthermore, we have used the notation in which arguments of dependent variables are replaced by subscripts (for example $U_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} = U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$). The explicit forms of U, X, Y and Z will be given later on in this note (see [23] for details). Eqs. (3), (4) have been derived originally in [24] and correspond to Eqs. (4.8), (4.9) in [23]. Sometimes these equations are known also as the Krasitskii equations.

The Higher Order Spectral Method. The starting point for introducing the HOS method is the following system of coupled evolution equations (see [4,5]):

$$\frac{\partial \eta}{\partial t} + \nabla_h \psi \nabla_h \eta - W\big(1 + (\nabla_h \eta)^2\big) = 0, \quad z = \eta(\mathbf{x}, t), \tag{5}$$

$$\frac{\partial \psi}{\partial t} + g\eta + \frac{1}{2}\big[(\nabla_h \psi)^2 - W^2\big(1 + (\nabla_h \eta)^2\big)\big] = 0, \quad z = \eta(\mathbf{x}, t), \tag{6}$$

where $W = \partial \phi / \partial z|_{z=\eta}$ is the vertical velocity computed on the free surface $\eta(\mathbf{x}, t)$ and ∇_h is the horizontal gradient operator. These two equations correspond to the dynamic and kinematic boundary conditions on the free surface for the Laplace equation. The major problem is to express the vertical velocity W as a function of ψ. To achieve such a task it is necessary to perform a series expansion for the velocity potential $\phi(\mathbf{x}, z, t)$ [5]:

$$\phi(\mathbf{x}, z, t) = \sum_{m=1}^{M} \phi^{(m)}(\mathbf{x}, z, t). \tag{7}$$

$\phi^{(m)}(\mathbf{x}, z, t)$ is assumed to be a quantity of order $O(\varepsilon^m)$, ε is a small parameter that represents physically a measure of steepness; M is the order of approximation of nonlinearity. The next step is to Taylor-expand each $\phi^{(m)}$ around $z = 0$. Collecting terms at each order, we obtain a sequence of boundary conditions for the unknowns $\phi^{(m)}$ on $z = 0$:

$$\phi^{(1)}(\mathbf{x}, 0, t) = \psi(\mathbf{x}, t), \tag{8}$$

$$\phi^{(m)}(\mathbf{x}, 0, t) = -\sum_{k=1}^{m-1} \frac{\eta^k}{k!} \frac{\partial^k}{\partial z^k} \phi^{(m-k)}(\mathbf{x}, 0, t) \quad (m = 2, 3, \ldots, M). \tag{9}$$

This method therefore consists in transporting the original problem for the unknown $\phi(\mathbf{x}, z, t)$ with complicated (and unknown) boundary $z = \eta(\mathbf{x}, t)$ to a sequence of M problems with unknowns $\phi^{(m)}(\mathbf{x}, z, t)$ with boundary given by $z = 0$. In infinite water depth, each $\phi^{(m)}(\mathbf{x}, z, t)$ is given by:

$$\phi^{(m)}(\mathbf{x}, z, t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{\Phi}^{(m)}(\mathbf{k}, t) \, e^{|\mathbf{k}|z} \, e^{i(\mathbf{k} \cdot \mathbf{x})} \, d\mathbf{k}. \tag{10}$$

Once $\phi^{(m)}$ are known, then it is possible to show that $W(\mathbf{x}, t)$ can be obtained in the following way:

$$W(\mathbf{x}, t) = \sum_{m=1}^{M} W^{(m)}(\mathbf{x}, t), \tag{11}$$

where $W^{(m)}(\mathbf{x}, t)$ is given by:

$$W^{(m)}(\mathbf{x}, t) = \sum_{k=0}^{m-1} \frac{\eta^k}{k!} \frac{\partial^{k+1}}{\partial z^{k+1}} \phi^{(m-k)}(\mathbf{x}, 0, t) \quad (m = 1, 2, \ldots, M). \tag{12}$$

As observed by Tanaka [10] and previously by West et al. [4], the substitution of W in the original equations (5), (6) should be done with some care, retaining the consistency of ordering with respect to the expansion parameter ε.

Relation between the two approaches. Consider Eqs. (5), (6) and rewrite them by inserting (12). Here, for simplicity we will consider $M = 3$, thus retaining the dynamics of four wave interactions; nonlinearity higher than cubic will be neglected, see [4,10]. Eqs. (5), (6) become:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi^{(1)}}{\partial z} + \left[-\nabla_h \psi \nabla_h \eta + \frac{\partial \phi^{(2)}}{\partial z} + \eta \frac{\partial^2 \phi^{(1)}}{\partial z^2} \right] + \left[\frac{\partial \phi^{(3)}}{\partial z} + \eta \frac{\partial^2 \phi^{(2)}}{\partial z^2} + \frac{\eta^2}{2} \frac{\partial^3 \phi^{(1)}}{\partial z^3} + \frac{\partial \phi^{(1)}}{\partial z} (\nabla_h \eta)^2 \right], \quad (13)$$

$$\frac{\partial \psi}{\partial t} = -g\eta + \left[-\frac{1}{2} (\nabla_h \psi)^2 + \frac{1}{2} \left(\frac{\partial \phi^{(1)}}{\partial z} \right)^2 \right] + \left[\frac{\partial \phi^{(1)}}{\partial z} \left(\frac{\partial \phi^{(2)}}{\partial z} + \eta \frac{\partial^2 \phi^{(1)}}{\partial z^2} \right) \right]. \quad (14)$$

The next step consists in using (8), (9) and (10) to express vertical derivatives of $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ as a function of ψ. The following results are obtained:

$$\frac{\partial \phi^{(1)}}{\partial z} = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} |\mathbf{k}| \hat{\psi}_{\mathbf{k}} \, e^{i(\mathbf{k} \cdot \mathbf{x})} \, d\mathbf{k}, \quad (15)$$

$$\frac{\partial \phi^{(2)}}{\partial z} = -\frac{1}{4\pi^2} \int\limits_{-\infty}^{+\infty} |\mathbf{k}_1| |\mathbf{k}_1 + \mathbf{k}_2| \hat{\psi}_{\mathbf{k}_1} \hat{\eta}_{\mathbf{k}_2} \, e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \, d\mathbf{k}_1 \, d\mathbf{k}_2, \quad (16)$$

$$\frac{\partial \phi^{(3)}}{\partial z} = \frac{1}{8\pi^3} \int\limits_{-\infty}^{+\infty} \left(|\mathbf{k}_1| |\mathbf{k}_1 + \mathbf{k}_2| |\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3| - \frac{1}{2} \left(|\mathbf{k}_1|^2 |\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3| \right) \right)$$
$$\times \hat{\psi}_{\mathbf{k}_1} \hat{\eta}_{\mathbf{k}_2} \hat{\eta}_{\mathbf{k}_3} \, e^{i(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \cdot \mathbf{x}} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3. \quad (17)$$

Substituting relations (15)–(17) into Eqs. (13), (14) and writing them in Fourier space, the following equations are obtained:

$$\frac{\partial \hat{\eta}_{\mathbf{k}}}{\partial t} = |\mathbf{k}| \hat{\psi}_{\mathbf{k}} + \int\limits_{-\infty}^{\infty} U'_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} \hat{\psi}_{\mathbf{k}_1} \hat{\eta}_{\mathbf{k}_2} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2} \, d\mathbf{k}_1 \, d\mathbf{k}_2$$
$$+ \int\limits_{-\infty}^{\infty} X'_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \hat{\psi}_{\mathbf{k}_1} \hat{\eta}_{\mathbf{k}_2} \hat{\eta}_{\mathbf{k}_3} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3, \quad (18)$$

$$\frac{\partial \hat{\psi}_{\mathbf{k}}}{\partial t} = -g\hat{\eta}_{\mathbf{k}} + \int\limits_{-\infty}^{\infty} Y'_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} \hat{\psi}_{\mathbf{k}_1} \hat{\psi}_{\mathbf{k}_2} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2} \, d\mathbf{k}_1 \, d\mathbf{k}_2$$
$$+ \int\limits_{-\infty}^{\infty} Z'_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \hat{\psi}_{\mathbf{k}_1} \hat{\psi}_{\mathbf{k}_2} \hat{\eta}_{\mathbf{k}_3} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3} \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3. \quad (19)$$

After some algebra, it is straightforward to show that:

$$U'_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} = U_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} = \frac{1}{2\pi} \left(\mathbf{k} \cdot \mathbf{k}_1 - |\mathbf{k}| |\mathbf{k}_1| \right),$$
$$Y'_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} = Y_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} = \frac{1}{4\pi} \left(\mathbf{k}_1 \cdot \mathbf{k}_2 + |\mathbf{k}_1| |\mathbf{k}_2| \right). \quad (20)$$

Therefore, kernels in the three wave interaction integrals obtained from HOS method coincide with the one reported by Krasitskii (see page 15 after Eq. (4.7) in paper [23]) obtained from the expansion of the Hamiltonian.

Kernels in the higher order integrals are given by:

$$X'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = \frac{1}{(2\pi)^2}\left(|\mathbf{k}||\mathbf{k}_1||\mathbf{k}-\mathbf{k}_3| - \frac{1}{2}|\mathbf{k}||\mathbf{k}_1|^2 - |\mathbf{k}_1||\mathbf{k}-\mathbf{k}_3|^2 + \frac{1}{2}|\mathbf{k}_1|^3 - |\mathbf{k}_1|\mathbf{k}_2\cdot\mathbf{k}_3\right),$$

$$Z'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = \frac{1}{(2\pi)^2}|\mathbf{k}_1||\mathbf{k}_2|\left(-|\mathbf{k}_2+\mathbf{k}_3| + |\mathbf{k}_2|\right). \tag{21}$$

They do not correspond to the kernels reported by Krasitskii [23]. Nevertheless, it should be noted that in order to maintain the Hamiltonian structure, the kernels should preserve some symmetries that arise from the fact that integrals are not affected by re-labeling the dummy integration variables. Therefore $X'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ and $Z'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ should be symmetrized, i.e., they should be substituted by the sum of their non-symmetrical transpositions, divided by the number of these transpositions. More explicitly $X'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ and $Z'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ should be replaced respectively by $\widetilde{X}'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$ and $\widetilde{Z}'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}$, given by (see also [25]):

$$\widetilde{X}'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = \frac{1}{2}\left(X'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} + X'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_2}\right),$$

$$\widetilde{Z}'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = \frac{1}{2}\left(Z'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} + Z'_{\mathbf{k},\mathbf{k}_2,\mathbf{k}_1\mathbf{k}_3}\right). \tag{22}$$

After this process and the use of the properties of the delta-function in the integrals, it turns out that:

$$\widetilde{X}'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = X_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = -\frac{1}{8\pi^2}|\mathbf{k}||\mathbf{k}_1|\left(|\mathbf{k}| + |\mathbf{k}_1| - |\mathbf{k}_1+\mathbf{k}_3| - |\mathbf{k}_1+\mathbf{k}_2|\right),$$

$$\widetilde{Z}'_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = Z_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = \frac{1}{8\pi^2}|\mathbf{k}_1||\mathbf{k}_2|\left(|\mathbf{k}_1| + |\mathbf{k}_2| - |\mathbf{k}-\mathbf{k}_2| - |\mathbf{k}-\mathbf{k}_1|\right), \tag{23}$$

i.e., kernels from the HOS method coincide with kernels in the dynamical equations (see [23]).

We have shown that one numerical method can be obtained from the other by just a proper symmetrization of the kernels in the integrals. The HOS method appears simpler with respect to the Hamiltonian expansion approach because it is much easier to include higher order terms in the simulation (just increase M using the recursive formula in (12)). Even though we have not proved it, we expect that for higher order terms ($M > 3$), the results just found will still hold. It also must be stressed that while the above methods are suitable for studying the statistical properties of the surface gravity waves, they are inadequate for studying the process of wave breaking. For this kind of problems different approaches should be considered (see for example [26]).

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